STOCHASTIC PROCESS – CONCEPTS & DEFINITIONS

- **Filtration** defined on a measurable space \((\Omega, \mathcal{F})\) is an \(\mathcal{F}\)-indexed set of \(\sigma\)-algebras \(\{\mathcal{F}_j\}, i \in \mathcal{I}\) that is ‘increasing’ and ‘becoming more complete’ in the sense that:

\[
\begin{align*}
\forall i, j \in \mathcal{I}, \quad i \leq j & \Rightarrow \mathcal{F}_i \subseteq \mathcal{F}_j \\
\forall i \in \mathcal{I}, \quad \mathcal{F}_i & \subseteq \mathcal{F}
\end{align*}
\]  

(163)

Each \(\sigma\)-algebra element of a filtration represents a ‘set of information’ or ‘state of knowledge’; filtration represents how information/knowledge ‘accumulates’.

- The **INDEX SET** \(\mathcal{I}\) may be discrete, e.g. \(\mathcal{I} = \{1,2,3\}\) or \(\mathcal{I} = \{0,1,2,\ldots\}\), as well as continuous, e.g. \(\mathcal{I} = [0, \infty)\) where the index \(t \geq 0\) often represents time.

For example, consider a measurable space created by two consecutive **coin toss** experiments:

\[
(\Omega, \mathcal{F}) \equiv \begin{cases} 
\Omega = \{HH, HT, TH, TT\} \\
\mathcal{F} = \Pi(\Omega) 
\end{cases}
\]  

(164)

One may then define three ‘increasing’ \(\sigma\)-algebras \(\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}\) as \(\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2\) thus:

\[
\begin{align*}
\mathcal{F}_0 &= \{\phi, \Omega\} \\
\mathcal{F}_1 &= \{\phi, \Omega, \{HH, HT\}, \{TH, TT\}\} \\
\mathcal{F}_2 &= \{\phi, \Omega, \{HH, HT\}, \{TH, TT\}, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \\
& \quad \{HT, TH, TT\}, \{HH, TH, TT\}, \{HH, HT, TT\}, \{HH, HT, TH\}\} = \Pi(\Omega)
\end{align*}
\]  

(165)
Clearly \( \{ \phi_i \}_{i=0}^2 \) forms a filtration, with the index designating how many times the coin has been tossed (with the result revealed).

Now suppose that the ‘physical’ probability that the coin would land ‘Head’ is given by the parameter \( p \in (0,1) \) for both tosses, and that events ‘1st toss’ and ‘2nd toss’ are independent, then the probability measure \( P \), induced as it were by \( p \), would furnish the probability space \( (\Omega, \varnothing, P) \), i.e. where:

\[
\begin{align*}
P(\emptyset) &= 0, P(\Omega) = 1, \\
P(\{HH,HT\}) &= p, P(\{TH,TT\}) = 1 - p, \\
P(\{HH\}) &= p^2, P(\{HT\}) = p(1 - p) = P(\{TH\}), P(\{TT\}) = (1 - p)^2, \\
P(\{HH,TH,TT\}) &= 2p(1 - p) + (1 - p)^2, \\
P(\{HH,TH,TT\}) &= p^2 + p(1 - p) + (1 - p)^2 = \{HH,HT,TT\}, \\
P(\{HH,TH,TT\}) &= p^2 + 2p(1 - p)
\end{align*}
\]

\section*{STOCHASTIC PROCESS, aka RANDOM PROCESS} \( \equiv \) a family of \( \mathcal{I} \)-indexed random variables, whence designated \( \{X_i\}_{i \in \mathcal{I}} \) or \( \{X_i\}, i \in \mathcal{I} \).

Alternatively, think of it as a measurable function \( X \) whose domain is the Cartesian product between the index set \( \mathcal{I} \) and the sample space \( \Omega \):

\[
X : \Omega \times \mathcal{I} \to \mathbb{R} \quad \text{where } X(i, \omega) = X_i(\omega), \quad i \in \mathcal{I}, \omega \in \Omega
\]

Conversely, or any given \( \omega \in \Omega \), the set \( \{X_i = X_{\omega}(i) = X(i, \omega)\}, i \in \mathcal{I} \) defines a \textit{PATH}, aka \textit{TRAJECTORY}, corresponding to \( \omega \).

Any stochastic process \( \{X_i\}, i \in \mathcal{I} \) is said to be \textit{ADAPTED} to the filtration \( \{\phi_i\}, i \in \mathcal{I} \) if \( \forall i \in \mathcal{I} \), the random variable \( X_i \) is \( \phi_i \)-measurable.

Often, the index represents \textit{the progress of time}, whence filtration represents \textit{an increasing information set} as time passes, and the stochastic process is naturally adapted in the sense that
one may not know/see the future; such a stochastic process may be designated \( \{X_t\}_{t \geq 0} \) or \( \{X_t, t \geq 0\} \), but more popularly \( \{X_t, t \geq 0\} \).

STOCHASTIC PROCESS – CLASSIFICATIONS & TYPES

- Characterising/categorising stochastic processes in terms of index set:
  - **DISCRETE(-TIME) STOCHASTIC PROCESS** \( \equiv \) a stochastic process whose index set is finite or countable. For example, \( \mathcal{I}_1 = \{0, 1, 2, \ldots\} \), \( \mathcal{I}_2 = \{2\pi, 4\pi, \ldots\} \), etc.
  - **CONTINUOUS(-TIME) STOCHASTIC PROCESS** \( \equiv \) a stochastic process whose index set is infinite. For example, \( \mathcal{I}_3 = (0, \infty) \), \( \mathcal{I}_4 = [0, 1]^3 \), etc.

- Characterising/categorising stochastic processes in terms of state space:
  - **DISCRETE-STATE (STOCHASTIC) PROCESS** \( \equiv \) a stochastic process whose random variables are not continuous functions on \( \Omega \) a.s.; in other words, the state space is finite or countable. So for each index value, \( X_i, i \in \mathcal{I} \) is a discrete r.v. with an associated p.m.f.
  - **CONTINUOUS-STATE (STOCHASTIC) PROCESS** \( \equiv \) a stochastic process whose random variables are continuous functions on \( \Omega \) a.s.; in other words, the state space is infinite. So for each index value, \( X_i, i \in \mathcal{I} \) is a continuous r.v. with an associated p.d.f.

- Characterising/categorising stochastic processes in terms of probabilistic property:
  - Over time?
  - Over increments?
  - Given the present state?
  - Given the past information?

- **STATIONARY PROCESS** \( \equiv \) a stochastic process \( \{X_t, t \geq 0\} \) whose probability distribution is invariant with time:

\[
F_{X_t}(x) = \Pr(X_t \leq x) = \Pr(X_s \leq x) = F_{X_s}(x), \quad 0 \leq s \leq t
\]

(168)
More generally, the time invariance property holds for joint distributions as well:

\[
F_{X_1,...,X_n}(x_1,...,x_n) = \Pr\{X_{t_1} \leq x_1, \ldots, X_{t_k} \leq x_k\} = \Pr\{X_{t_1+h} \leq x_1, \ldots, X_{t_k+h} \leq x_k\} = F_{X_{t_1+h},...,X_{t_k+h}}(x_1,...,x_n), \quad h, t_1, \ldots, t_k \geq 0
\]

(169)

Often, the original process isn’t stationary, but for a fixed interval \(\Delta t > 0\), the

**INCREMENT** defined by \(\{X_{t+\Delta t} - X_t, t \geq 0\}\) is (a stationary stochastic process).

- **INDEPENDENT INCREMENT PROCESS**, aka **process with independent increments** \(\equiv\) a stochastic process \(\{X_t, t \geq 0\}\) any pair of whose non-overlapping increments (which need not be over the same interval length) are independent:

\[
F_{(X_s-X_u),(X_t-X_v)}(a,b) = \Pr\{(X_s-X_u) \leq a \cap (X_t-X_v) \leq b\} = \Pr(X_s-X_u \leq a) \cdot \Pr(X_t-X_v \leq b) = F_{(X_s-X_u)}(a) \cdot F_{(X_t-X_v)}(b), \quad 0 \leq s < t < u < v
\]

(170)

- **STATIONARY INDEPENDENT INCREMENT PROCESS**, aka **process with stationary independent increments** \(\equiv\) a stochastic process whose non-overlapping increments are **i.i.d.**

- **MARTINGALE** \(\equiv\) a stochastic process \(\{X_t, t \geq 0\}\) whose expectation in the future is finite and equal to the present state:

\[
E[X_t] < \infty, \quad t \geq 0
\]

\[
E\left[X_u \mid x_{t_1}, \ldots, x_{t_n}\right] = x_{t_n}, \quad 0 \leq t_1 < \cdots < t_n < u
\]

(171)

- **SUBMARTINGALE** \(\equiv\) a stochastic process \(\{X_t, t \geq 0\}\) whose expectation in the future is finite and **greater than or equal to** the present state:

\[
E[X_t] < \infty, \quad t \geq 0
\]

\[
E\left[X_u \mid x_{t_1}, \ldots, x_{t_n}\right] \geq x_{t_n}, \quad 0 \leq t_1 < \cdots < t_n < u
\]

(172)

- **SUPERMARTINGALE** \(\equiv\) a stochastic process \(\{X_t, t \geq 0\}\) whose expectation in the future is finite and **less than or equal to** the present state:
\[ E \left[ X_t \right] < \infty, \quad t \geq 0 \]
\[ E \left[ X_u \left\{ x_{t_1}, \ldots, x_{t_n} \right\} \right] \leq x_u, \quad 0 \leq t_1 < \cdots < t_n < u \]  

(173)

Hence a martingale is *simultaneously* both a submartingale and a supermartingale.

STOCHASTIC PROCESS – MARKOV PROCESSES

- **MARKOV PROCESS** ≡ a stochastic process \( \{ X_t, t \geq 0 \} \) with **MARKOV PROPERTY**, i.e. that the probability distribution of future state(s) conditional to revealed states (*i.e. the current state of knowledge, accumulating all information from the past up to the present*) is only a function of the present (most recently revealed) state:

\[
P_{X_t | x_{t_1}, \ldots, x_{t_n}} (x) = \Pr \left( X_u \leq x \left| x_{t_1}, \ldots, x_{t_n} \right. \right)
= \Pr \left( X_u \leq x \right| x_{t_n} \right) = P_{X_{t_n}} (x), \quad 0 \leq t_1 < \cdots < t_n < u
\]

(174)

- **MARKOV CHAIN** ≡ a discrete(-time) Markov process.

A Markov chain can be discrete/continuous w.r.t. time as well as w.r.t. state.

- **FINITE-DIMENSIONAL MARKOV CHAIN** ≡ a Markov chain with finite, constant number \( n \), \( 2 \leq n < \infty \) of states, generally represented quite conveniently with:

  (i) sequentially realized **STATE VECTORS**:

\[
s_t \in \left\{ x \in \{0,1\}^n \left| x = x_1 + \cdots + x_n = 1 \right. \right\}
= \left\{ \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
\end{array} \right\} = \{ e_1, \ldots, e_j, \ldots, e_n \}, \quad t \geq 0
\]

(175)

(ii) **(STATE) PROBABILITY VECTORS** (n/a at time \( t = 0 \)):
\[ p_t \in \{ x \in [0,1]^n \mid ||x|| = x_1 + \cdots + x_n = 1 \} , \quad t > 0 \]  

(iii) and (square) (STATE) TRANSITION PROBABILITY MATRICES:

\[
M_t = \begin{pmatrix}
    p_{1e-1} & \cdots & p_{1e-j} & \cdots & p_{1e-n} \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    p_{ne-1} & \cdots & p_{ne-j} & \cdots & p_{ne-n}
\end{pmatrix} \in [0,1]^{n \times n}
\]  

\( \exists \forall j \in \{1, \ldots, n\}, \quad \langle M_t \rangle_{ij} = \sum_{i=1}^{n} \langle M_t \rangle_{ij} = 1, \quad t \geq 0 \)

So that at any time \( t \geq 0 \):

(conditionally) \( p_{t+1} | s_t = M_s p_t \)

(unconditionally) \( p_{t+1} = M_p p_t \) \hspace{1cm} (178)

In a sense, a state vector can be seen as a state ‘probability’ vector \textit{once randomness is removed}. Indeed, it’s possible to define the state probability vector at time \( t = 0 \), provided it is equated to the initial, TRANSIENT STATE (vector): \( p_0 = s_0 \).

\[
p_t = M_{t-1} p_{t-1} = M_{t-2} M_{t-2} p_{t-2} = \cdots = \prod_{i=1}^{t} M_{i-1} p_0
\]  

(TIME-)HOMOGENEOUS MARKOV CHAIN, aka STATIONARY MARKOV CHAIN \equiv a Markov chain with a constant transition probability matrix:

\( \forall t \in \{0,1,2,\ldots\}, \quad M_t = M \Rightarrow p_t = M p_{t-1} = M^2 p_{t-2} = \cdots = M^t p_0 \) \hspace{1cm} (180)

Eventually, such a Markov chain settles down to some STEADY STATE (probability vector):

\[ p = M p = \lim_{t \to \infty} M^t s_0 \]  

ABSORBING STATE \equiv any state \( j \in \{1, \ldots, n\} \) with zero outgoing probability:
\[ j \ni \langle M \rangle_{ij} = \begin{bmatrix} \langle M \rangle_{1j} \\ \vdots \\ \langle M \rangle_{nj} \end{bmatrix} = e_j \] (182)

- **ACCESSIBLE STATE** defined w.r.t. a current state \( j \in \{1, \ldots, n\} \equiv \) any state \( i \in \{1, \ldots, n\} \) which can be reached in some \( t \geq 1 \) periods:

\[ i \ni \langle M \rangle_{ij}^t > 0 \quad (183) \]

- **COMMUNICATING CLASS** \( C \subseteq \{1, \ldots, n\} \) any of whose pair of states \( i, j \in C\) COMMUNICATE with one another, written \( i \leftrightarrow j \), meaning that one is accessible to another and vice versa, but with no other states outside the class:

\[ C \subseteq \{1, \ldots, n\} \ni \forall i, j \in C, \quad i \leftrightarrow j \\
\exists k \in \{1, \ldots, n\} \setminus C \ni i \leftrightarrow k \quad (184) \]

where \( i \leftrightarrow j \Leftrightarrow \exists u, v \geq 1 \ni \langle M^u \rangle_{ij}^v > 0 \text{ and } \langle M^u \rangle_{ji}^v > 0 \]

- **IRREDUCIBLE MARKOV CHAIN** \( \equiv \) a Markov chain whose state space forms a communicating class

- For example, consider a 4-state time-homogeneous/stationary Markov chain:

\[ M = \begin{pmatrix} p_{1e-1} & p_{1e-2} & p_{1e-3} & p_{1e-4} \\ p_{2e-1} & p_{2e-2} & p_{2e-3} & p_{2e-4} \\ p_{3e-1} & p_{3e-2} & p_{3e-3} & p_{3e-4} \\ p_{4e-1} & p_{4e-2} & p_{4e-3} & p_{4e-4} \end{pmatrix} \in [0,1]^4 \quad (185) \]

- Also, introduce the notion of adjacency, i.e. in the sense that the for each of the 3 non-absorbing states, the probabilities of not transiting to other states are the highest, followed by
the probabilities of transiting to adjacent states, which in turns are higher than the
probabilities of transiting to the nearest non-adjacent states, and so on, for example:

\[
M = \begin{pmatrix}
0.80 & 0.08 & 0.04 & 0.02 \\
0.10 & 0.80 & 0.08 & 0.08 \\
0.08 & 0.08 & 0.80 & 0.10 \\
0.02 & 0.04 & 0.08 & 0.80 \\
\end{pmatrix}
\]  \hspace{1cm} (186)

- Multiplying this matrix by itself just 20 times yields:

\[
M^{20} = \begin{pmatrix}
0.2 & 0.2 & 0.2 & 0.2 \\
0.3 & 0.3 & 0.3 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.3 \\
0.2 & 0.2 & 0.2 & 0.2 \\
\end{pmatrix}
\]  \hspace{1cm} (187)

- Clearly, each and every column converges to (become) the \textit{steady state probability vector} \( \mathbf{p} \).

- Now, let the 4\textsuperscript{th} state be \textit{absorbing}, hence the transition probability matrix of the form:

\[
\tilde{M} = \begin{pmatrix}
p_{1\to1} & p_{1\to2} & p_{1\to3} & 0 \\
p_{2\to1} & p_{2\to2} & p_{2\to3} & 0 \\
p_{3\to1} & p_{3\to2} & p_{3\to3} & 0 \\
p_{4\to1} & p_{4\to2} & p_{4\to3} & 1 \\
\end{pmatrix}
\]  \hspace{1cm} (188)

- In particular, continuing with the above example, consider:

\[
\tilde{M} = \begin{pmatrix}
0.80 & 0.08 & 0.04 & 0 \\
0.10 & 0.80 & 0.08 & 0 \\
0.08 & 0.08 & 0.80 & 0 \\
0.02 & 0.04 & 0.08 & 1 \\
\end{pmatrix}
\]  \hspace{1cm} (189)

- Multiplying this matrix by itself 50 times yields:
Eventually, each and every column converges to a **DEGENERATE STATE PROBABILITY VECTOR**, namely $\mathbf{p} = \mathbf{e}_4$ in this example, meaning that probabilistically speaking, the Markov chain settles down to the absorbing 4th state—in this example less than 100 iterations were required to get the convergence down to within a *basis point* away from getting perfect zeros and ones, *whence no further state migration is ever likely to happen!*

**FINITE-DIMENSIONAL CONTINUOUS-TIME MARKOV PROCESS** = a continuous time Markov process with finite, constant number $n$, $2 \leq n < \infty$ of states, where instead of working directly with a state transition probability matrix, it’s better (more convenient) to work through a **STATE TRANSITION INTENSITY MATRIX**:

$$
Q = \begin{pmatrix}
q_{1e^{-1}} & \ldots & q_{1e^{-j}} & \ldots & q_{1e^{-n}} \\
\vdots & \ddots & \vdots & & \vdots \\
q_{ne^{-1}} & \ldots & q_{ne^{-j}} & \ldots & q_{ne^{-n}} \\
\end{pmatrix} 
\in [0,1]^{n \times n}
$$

$$
\exists \forall j \in \{1, \ldots, n\}, \quad q_{je^{-j}} = -\sum_{i \neq j}^{n} q_{ie^{-j}}
$$

**WAITING TIME** = time spent in a particular state $j \in \{1, \ldots, n\}$ (having thus entered) before exiting (transiting to another state); it is an exponentially distributed with parameter $\lambda_j \equiv q_{je-j}$ given by the diagonal entry corresponding to that state:

$$
\Pr(W_j \leq w) = 1 - e^{-\lambda_j w}, \quad j = 1, \ldots, n
$$
STOCHASTIC PROCESS – RANDOM WALKS

- **RANDOM WALK** ≡ a mathematical model, in the form of a discrete stochastic process \( \{X_t, t \geq 0\} \), describing any physical/simulated/conceptual ‘trajectory’ that arises from taking sequential equal-sized ‘steps’ (in the state space) which are random, and independently so, in directions.
  - **SYMMETRIC RANDOM WALK** ≡ a random walk with equal probability in taking steps in all the available directions.

- **(STANDARD) BROWNIAN MOTION** ≡ a continuous stochastic process \( \{X_t, X \geq 0\} \) — named in honour of *Robert Brown*, the Scottish botanist who is generally accredited with discovering the phenomena associated with random physical movements of particles suspended in a fluid—with the following properties:
  - It has a *continuous trajectory*, which begins at \( X_0 = 0 \).
  - It has *stationary independent increments* (non-overlapping and of equal interval length).
  - In terms of *probability law*, the probability measure is defined such that any increment is *normally distributed* with variances proportional to interval length:
    \[
    \forall s, t, s \leq t, \quad X_t - X_s \sim N(0, t - s)
    \]  
    (193)
  - So that, after some time interval \( \Delta t > 0 \), the probability of finding the *trajectory point* anywhere up to location \( a \in \mathbb{R} \) is given by:
    \[
    \Pr(X_t \leq a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi\Delta t}} \exp\left(-\frac{x^2}{2\Delta t}\right) dx
    \]  
    (194)

- **(STANDARD) WIENER PROCESS** ≡ a continuous stochastic process \( \{X_t, X \geq 0\} \) — named in honour of *Norbert Wiener*, the American cyberneticist (although he himself was said to prefer the term *cybernetician*)—with the following properties:
  - It has a continuous trajectory, which begins at \( X_0 = 0 \).
It is a square integrable martingale, where:
\[ \forall s, t, s \leq t, \quad E[(W_t - W_s)^2] = t - s \] (195)

It’s been shown, by the French mathematician Paul Pierre Levy, that:

*Mutatis mutandis a Wiener process is mathematically equivalent to a Brownian motion.*

- The Brownian motion/Wiener process \( \{X_t, X \geq 0\} \) can be seen as a *limit in distribution* of a symmetric random walk:
  - Divide time period \([0,T]\) into \( n > 1 \) intervals, each of length \( \Delta = \frac{T}{n} \).
  - As with BM, let \( X_0 = 0 \).
  - Then take a step increment \( S_1 \) of size \( \sqrt{\Delta} \) is taken in either direction (plus or minus) with equal probabilities \( p = \frac{1}{2} \), in other words, \( S_1 \) is a ‘uniform’ *Bernoulli r.v. rescaled* from \([0,1]\) to \([\pm \sqrt{\Delta}, \pm \sqrt{\Delta}]\):

\[
B_t \sim Bernoulli(p = \frac{1}{2}) \Rightarrow \begin{cases} 
\Pr(S_1 = s) = \begin{cases} 
\frac{1}{2} & s = \pm \sqrt{\Delta} \\
0 & \text{otherwise}
\end{cases} \\
E[S_1] = p = \frac{1}{2}, \quad \text{Var}(S_1) = p(1 - p) = \frac{1}{4}
\end{cases}
\]

\( X_\Delta = X_0 + S_1 \)

- Thereafter, i.e. for \( t = 2\Delta, 3\Delta, ..., T - \Delta \), the process advances randomly in the same manner but *independently* of the previous step, i.e. probability of either direction still remains at \( p = \frac{1}{2} \), regardless of whichever direction the prior step went, so that all together:

\[
\begin{align*}
(i) & \quad X_0 = 0 \\
(ii) & \quad S_i = \sqrt{\Delta} \cdot (2B_i - 1), \quad B_i \sim Bernoulli(p = \frac{1}{2}), \quad i = 1, \ldots, n \\
(iii) & \quad E[S_i, S_j] = E[S_i] \cdot E[S_j], \quad i, j \in \{1, \ldots, n\}, \quad i \neq j \\
(iv) & \quad X_{t+\Delta} = X_t + S_i, \quad t = \Delta \cdot (i - 1), \quad i = 1, \ldots, n
\end{align*}
\] (197)

- In other words, \( X_f \) is recognisably a *Binomial random variable.*
\[ X_T = \sum_{i=1}^{n} S_i = \sum_{i=1}^{n} \left( \sqrt{\Delta} \cdot (2B_i - 1) \right) \]
\[ = 2\sqrt{\Delta} \cdot \left( \sum_{i=1}^{n} B_i \right) - n\sqrt{\Delta} \]
\[ = 2\sqrt{\Delta} \cdot B - n\sqrt{\Delta} , \quad B \sim Bin(n, p = \frac{1}{2}) \]  

Therefore, in terms of 1st and 2nd moment statistics:

\[
E[X_T] = 2\sqrt{\Delta} \cdot E[B] - n\sqrt{\Delta} = 2\sqrt{\Delta} \cdot np - n\sqrt{\Delta} = 2\sqrt{\Delta} \cdot \frac{n}{2} - n\sqrt{\Delta} = 0
\]

\[
Var(X_T) = \left(2\sqrt{\Delta}\right)^2 \cdot Var(B) = 4\Delta np(1 - p) = 4\Delta n \frac{1}{2} \left(1 - \frac{1}{2}\right) = \Delta n = T
\]  

Citing the Central Limit Theorem (C.L.T.), as the time partition gets infinitely finer, \( n \to \infty \), the random variable \( X_T \) converges in distribution (i.e. convergence in law) to a Normally distributed random variable:

\[
\lim_{n \to \infty} (X_T = 2\sqrt{\Delta} \cdot B - n\sqrt{\Delta}) \sim N(0, T)
\]  

Of course, one can always add up such independent random variables, i.e.

\( X_{T_i + T_j} = X_{T_i} + X_{T_j} \), or carve out any smaller time interval, i.e. \( X_{t_i} , t \in (0, T) \), or extend time to infinity, i.e. \( T \to \infty \), resulting in a proper Brownian motion/Wiener process.
STOCHASTIC PROCESS – COUNTING PROCESSES

- **COUNTING PROCESS** ≡ a stochastic process \( \{N_t, t \geq 0\} \) whose states are non-negative integers, non-decreasing, and right-continuous:
  
  \[
  \begin{align*}
  (i) \quad & N_t \in \{0,1,2,\ldots\}, \quad t \geq 0 \\
  (ii) \quad & s < t \Rightarrow N_s \leq N_t \\
  (iii) \quad & N_t = \lim_{h \to 0^+} N_{t+h} = N_t \\
  (iv) \quad & E[N_t] < \infty, \quad 0 \leq t < \infty
  \end{align*}
  \]

- The counting process could be restricted further by the condition that the value can only increase by one unit at a time:
  \[
  N_{t^+} - N_t \in \{0,1\}, \quad N_t = \lim_{h \to 0^+} N_{t+h} = N_t
  \]

- For such a counting process, it’s also possible to define/associate another stochastic process designating the **INTERARRIVAL TIMES** \( \{A_k, k > 0\} \), aka simply as interarrivals:
  
  \[
  A_k = \min\{t \mid N_t = 1\} \\
  A_k = \min\{t \mid N_t = k\} - A_{k-1}, \quad k = 2,3,\ldots
  \]

- **RENEWAL PROCESS** ≡ a counting process \( \{N_t, t \geq 0\} \) with i.i.d. interarrival times.

- **POISSON PROCESS** ≡ a counting process with Poisson probability distribution.

- Recall that:
  \[
  X \sim \text{Poisson}(\lambda) \quad \Rightarrow \quad \Pr(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad \Rightarrow \quad E[X] = Var(X) = \lambda
  \]

- **(TIME-)HOMOGENEOUS POISSON PROCESS**, aka **STATIONARY POISSON PROCESS** ≡ a Poisson process with constant INENSITY PARAMETER \( \lambda > 0 \):
\[ N_0 = 0 \]
\[ N_t \sim \text{Poisson}(\lambda t) \Rightarrow \Pr(N_t = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots \quad (205) \]

- Here it turns out that the associated interarrival times \( \{A_k, k > 0\} \) are i.i.d.r.v.'s with an Exponential distribution; in other words, a homogeneous Poisson process is a special case of a renewal process specifying \textit{exponentially distributed} interarrival times:
\[ N_t \sim \text{Poisson}(\lambda t) \Rightarrow A_k \sim \text{Exp}(\lambda), \quad \text{i.i.d.r.v.'s } X_1, X_2, \ldots \quad (206) \]

- Recall that:
\[ X \sim \text{Exp}(\lambda) \Rightarrow \Pr(X \leq x) = 1 - e^{-\lambda x}, \quad x \geq 0 \Rightarrow \quad \mathbb{E}[X] = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2} \quad (207) \]

- \textbf{(TIME-)INHOMOGENEOUS POISSON PROCESS, aka NON-STATIONARY POISSON PROCESS} \( \equiv \) a Poisson process with an \textit{INTENSITY FUNCTION} (of time)
\[ \lambda(t) > 0, \quad t \geq 0: \]
\[ N_0 = 0 \]
\[ \Lambda(t) \equiv \int_0^t \lambda(s) ds \]
\[ \Pr(N_t = n) = \frac{e^{-\Lambda(t)} (\Lambda(t))^n}{n!}, \quad n = 0, 1, 2, \ldots \quad (208) \]

- **COX PROCESS, aka DOUBLY STOCHASTIC POISSON PROCESS** \( \equiv \) a Poisson process whose intensity is itself an independent stochastic process.

- **COMPOUND POISSON PROCESS** \( \equiv \) a sum \( \{S_t, t \geq 0\} \) of \( N_t \) i.i.d.r.v.'s where \( \{N_t, t \geq 0\} \) is itself an independent (generally homogeneous/stationary) Poisson process:
\[ S_i = \sum_{n=1}^{N_i} X_i, \quad \text{i.i.d.r.v.'s } X_1, \ldots, X_{N_i} \]

\[ \Pr(N_i = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots \] (209)

\[ \square \] Independence, in turn, ensures that, in terms of expectation:

\[ E[X_i] = \mu \Rightarrow E[S_i] = E[E[S_i | N_i]] : conditional \ expectation \]

\[ = E[N_i E[X_i]] : X_i \text{ i.i.d.r.v.'s} \]

\[ = E[N_i] E[X_i] : independence \ assumption \]

\[ = \lambda t \mu : N_i \sim \text{Poisson}(\lambda t) \] (210)

\[ \square \] Similarly, for variance:

\[ \text{Var}(X_i) = \sigma^2 \Rightarrow \text{Var}(S_i) = E[\text{Var}(S_i | N_i)] + \text{Var}(E[S_i | N_i]) \]

\[ = E[N_i \text{Var}(X_i)] + \text{Var}(N_i E[X_i]) \]

\[ = \text{Var}(X_i) E[N_i] + E[X_i]^2 \text{Var}(N_i) \]

\[ = \lambda t \text{Var}(X_i) + E[X_i]^2 \] (211)