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ประเด็นศึกษาด้านวิศวกรรมการเงิน”
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[Pricing Multi-Asset/Basket-Referenced Derivatives via Copulas]
Outline of the Presentation

I. Introduction
   - What’s a copula? Why copulas (in financial modelling)?
   - What are basket-referenced/multi-asset/multiname (credit) derivatives?

II. Methodology
   - Introduction to copulas, particularly the elliptical (e.g. Gaussian) copulas
   - Pricing credit derivatives via Monte Carlo simulation

III. Examples
   - Gaussian copula – pricing n-th-to-default CDS & CDO
   - Gaussian slug copula – results from US & Thai equity data

IV. Q&A
   - Is it really “Recipe for Disaster: The Formula That Killed Wall Street”?
I. Introduction

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I. Introduction – what’s a copula?

- **Let’s talk mathematics.**
  - What kind of mathematical object is it?
    - A copula is a function from a hypercube to unit interval, $C: [0,1]^n \rightarrow [0,1]$, with some special properties [http://en.wikipedia.org/wiki/Copula_(statistics)].
    - Alternatively, a copula is a functional, i.e. a function of functions.
  - What does it do?
    - Recall the definitions: a univariate (cumulative) distribution function (c.d.f.), $F(x) \equiv \Pr(X \leq x)$, as well as a bivariate joint c.d.f., $F(x,y) \equiv \Pr(X \leq x \cap Y \leq y)$.
    - A bivariate copula takes 2 univariate c.d.f.’s, i.e. marginal distributions/marginals, $F_X(x) \equiv \Pr(X \leq x)$ & $F_Y(y) \equiv \Pr(Y \leq y)$ to make 1 bivariate joint c.d.f.: $C(F_X(x), F_Y(y)) = F(x,y)$.
    - To put it the other way around: a bivariate copula decomposes a bivariate joint c.d.f. into (a function of) 2 univariate c.d.f.’s — decomposition unique in case of continuous c.d.f.’s.
  - How so? **Sklar’s theorem (1959)**
I. Introduction – why copulas?

- Let’s talk probabilistic modelling.

  - Wouldn’t it be nice to be able to generalise the notion of “correlation”?
    - Whereas (statistical) independence, \( F(x, y) = F(x)F(y) \), is a singular concept, as are perfect “co-movements”, intermediate cases aren’t, and need to be defined.
    - In fact, the ubiquitous, much loved/abused, Pearson (product-moment) correlation, \( \rho \in [0,1] \), defined in terms of joint expectation, \( E[(X-\mu_X)(Y-\mu_Y) / (\sigma_X\sigma_Y)] \), is but one characterisation of “co-moving” amongst many definitions that are possible.

  - Wouldn’t it be nice to be able to model random multivariate in stages?
    (i) First model the individual random variables separately.
    (ii) Then later model how they “co-move” together.
    - With copula, not only can we break down the problem into 2 stages, we can even go ahead and perform (ii) alongside, or even ahead of, carrying out (i) first!
I. Introduction – why copulas? (2)

- Let’s talk risks.
  - Because individually, *risk drivers*,
    i.e. general (financial-economic) *risk factors*
    together w/ specific (portfolio/unhedged) *risk exposures*,
    - can be discrete,
    - are not *normally distributed*,
    - and/or otw. don’t follow *Brownian motions*.
  - Because collectively, risk drivers exhibit more general dependence structure
    - beside that which coresponds to ‘Pearson’s rho’,
      and may exhibit *asymmetry*, i.e. ‘Black Monday’,
      as well as *nonlinearity*, i.e. diminishing sensitivity.
  - Because we wish to capture *tail dependence*: \( \lim_{s \to \infty} \Pr(Y > s / X > s) \)
Let’s talk financial models.

- Multi(bi)variate normal distribution as a ‘packaged’ foundation
  - Normally distributed marginals ⇒ the ‘mu’ & ‘sigma’ vectors
  - Linear (Pearson’s) correlation ⇒ the ‘Rho’ matrix, hence our familiarity w/ such terms as positive semi-definiteness, quadratic form, Cholesky decomposition ...

- of modern finance:
  - Modern Portfolio Theory (MPT) ⇒ investment management
  - Capital Asset Pricing Model (CAPM) ⇒ financial econometrics
  - Asymptotic Single Risk Factor (ASRF) ⇒ Basel II capital standard

- If only we could …
  (i) decompose the multivariate normal distribution into a Gaussian copula + the normal marginals, then
  (ii) allow for other, i.e. non-normal, marginals?
I. Introduction – what are … derivatives?

- Let’s talk *basket-referenced/multi-asset/multiname (credit) derivatives*.
  - What is a *Credit Default Swaps (CDS)*?
    - A CDS is essentially an insurance policy written on a defined *credit event*, especially an *(obligor) default* (on loan/debt instrument).
  - What is a *1st-to-default CDS (1tD-CDS)*?
    - Likewise what are *2nd-to-default CDS (2tD-CDS), nth–to–default CDS (NtD-CDS)*?
  - What are *Collateralized Debt Obligations (CDO)*?
    - A *securitisation* device.
    - A *tranche* structure: e.g. with the *equity tranche* absorbing the first [0%-3%] of loss vis-à-vis the “collateral/asset pool”, and *mezzanine tranche, senior tranche, and super senior* absorbing, respectively, the (3%-10%), (10%-30%), and (30%-100%) portions thereafter.
  - With 1tD-CDS & CDO, pricing is effected by the degree to which individual defaults (random events) aren’t totally independent, but tied together by risk factors.
II. Methodology

...
II. Methodology – introducing Gaussian copulas

- Let’s begin with bivariate copulas
  - The notion of concordance order
    
    \[
    C_1 < C_2 \iff \forall u, v \in [0,1], \quad C_1(u, v) \leq C_2(u, v) \\
    C_1 > C_2 \iff \forall u, v \in [0,1], \quad C_1(u, v) \geq C_2(u, v)
    \]

- 3 special (fundamental/basic) copulas
  - The independent copula, \( C_i(u, v) = uv \), expresses statistical independence.
  - The minimum copula, \( C^- (u, v) = \max\{0, u + v - 1\} \), as well as the maximum copula, \( C^+ (u, v) = \max\{u, v\} \), expresses perfect co-movement (in either direction).

\[
\forall u, v \in [0,1], \\
C^-(u, v) \equiv \max\{0, u + v - 1\} \leq C(u, v) \leq C^+(u, v) \equiv \min\{u, v\}
\]

- Also, just as a continuous c.d.f. can be differentiated, resulting in a probability density function (p.d.f.), so too can we work with a copula density function.

\[
c(u, v) \equiv \frac{\partial^2 C(u, v)}{\partial u \partial v}
\]
II. Methodology – introducing Gaussian copulas (2)

- How many (kinds of) copulas are (out) there?
  - What makes a bivariate copula?
    - $C(u,0) = \Pr(U \leq u \cap V \leq 0) = 0$
    - $C(1,v) = \Pr(U \leq 1 \cap V \leq v) = v$
    - $C$ is 2-increasing.

\[
\begin{align*}
0 \leq u_1 \leq u_2 \leq 1 \\
0 \leq v_1 \leq v_2 \leq 1 \\
\Rightarrow \quad C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0
\end{align*}
\]

- What is a parametric copula family?
  - The idea is to use a parameter to tune a degree of dependency.

- What about so-called elliptical copulas?
  - The idea is to borrow the dependency structure from any elliptical distribution.
  - Indeed the Gaussian copula is an elliptical copula, so is the Student’s t copula.
II. Methodology – introducing Gaussian copulas (3)

- The “3 islands” analogy
  - The ‘Gauss’ island …
    ... is populated by 2 (correlated) standard normal random variables $Z_1$ and $Z_2$.
  - The ‘Copula’ island …
    ... is populated by 2 uniform random variables $U_1$ and $U_2$.
  - The ‘General’ island …
    ... is populated by 2 random variables $X$ and $Y$.

\[
\begin{align*}
\{U_1 \leq u_1\} & \iff \{X \leq x\} \iff \{Z_1 \leq z_1\} \\
\{U_2 \leq u_2\} & \iff \{Y \leq y\} \iff \{Z_2 \leq z_2\}
\end{align*}
\]

These random variables are tied together...
...in the sense that...
...these events are considered equivalent.

\[
\{U_1 \leq u_1 \land U_2 \leq u_2\} \iff \{X \leq x \land Y \leq y\} \iff \{Z_1 \leq z_1 \land Z_2 \leq z_2\}
\]
II. Methodology – introducing Gaussian copulas (4)

- The transformation (transportation between the “3 islands”)

\[
\begin{align*}
\mathbf{u}_1, \mathbf{x}, \mathbf{z}_1 \quad \Rightarrow \quad & u_1 = F_X(x) = \Phi(z_1) \\
\mathbf{u}_2, \mathbf{y}, \mathbf{z}_2 \quad \Rightarrow \quad & u_2 = F_Y(y) = \Phi(z_2)
\end{align*}
\]

\[
\begin{align*}
F_X^{-1}(u_1) &= x = F_X^{-1}(\Phi(z_1)) \\
\Phi^{-1}(u_1) &= \Phi^{-1}(F_X(x)) = z_1 \\
F_Y^{-1}(u_2) &= y = F_Y^{-1}(\Phi(z_2)) \\
\Phi^{-1}(u_2) &= \Phi^{-1}(F_Y(y)) = z_2
\end{align*}
\]

- Hence the following joint c.d.f. definitions are kept identical.

<table>
<thead>
<tr>
<th>joint uniform distribution</th>
<th>bivariate normal (Gaussian)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Pr(\mathbf{u}_1 \leq \mathbf{u}_1 \cap \mathbf{u}_2 \leq \mathbf{u}_2) )</td>
<td>( \Pr(\mathbf{Z}_1 \leq z_1 \cap \mathbf{Z}_2 \leq z_2) )</td>
</tr>
<tr>
<td>( = \Pr(\mathbf{Z}_1 \leq \Phi^{-1}(\mathbf{u}_1) \cap \mathbf{Z}_2 \leq \Phi^{-1}(\mathbf{u}_2)) )</td>
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expressed as a function over \([0,1]^2\)

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</tbody>
</table>

expressed as a function of 2 c.d.f.
Finally ...

**interpretation 1:** bivariate c.d.f. over unit-square support, i.e. with uniform marginals

\[
C : [0,1]^2 \rightarrow [0,1]
\]

\[
C(u_1, u_2) = \Phi\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2) \bigg| \rho \right) = \Pr(U_1 \leq u_1 \cap U_2 \leq u_2)
\]

**interpretation 2:** functional that takes two univariate c.d.f.'s to produce a joint c.d.f.

\[
C : D \subseteq \mathbb{R}^2 \rightarrow [0,1]
\]

\[
C(x, y) = C\left((F_X(x)), (F_Y(y))\right) = \Phi\left(\Phi^{-1}(F_X(x)), \Phi^{-1}(F_Y(y)) \bigg| \rho \right) = \Pr(X \leq x \cap Y \leq y)
\]
II. Methodology – introducing Gaussian copulas (6)

- This is what a bivariate Gaussian copula density looks like:
II. Methodology – how do we use Gaussian copulas

- Let’s generalise from bivariate to “truly” multivariate copulas.
  - As with multivariate normal distributions, the parameterisation of a multivariate Gaussian copula is achieved by way of a matrix, i.e. the *correlation matrix*.

- Application conundrum.
  - Fact …
    - A standard normal p.d.f. cannot be analytically integrated.
    - There are many other copulas that don’t require numerical integrations.
  - Then …
    - Why is the Gaussian copula so popular in financial application?
    - Is it because financial analysts grew up on Markowitz’ MPT?
    - More likely, it’s because Gaussian copula is *(Monte Carlo) Simulation* friendly.
    - And nowadays complex financial derivatives are priced exactly via such a method.
  - Hence …
II. Methodology – how do we use Gaussian copulas (2)

- Let’s consider a “Monte Carlo expectation” algorithm.
  - Recall how $E[g(X)]$ can be approximated by taking the average of the values of $g$ evaluated at the many many many pseudo-randomly generated random variates.
  - To generate $E[g(X_1, \ldots, X_n)]$, where each individual $X_i$ can have any distribution (e.g. $X_1$ is exponentially distributed while $X_2$ is normally distributed, and so on), with dependency structure specified as a Gaussian copula (i.e. parameterised by the correlation matrix $R$), for each simulation run:
    (i) Generate $n$ i.i.d. pseudo-random standard normal random variates $\{z_1, \ldots, z_n\}$ ➔ vector $z$.
    (ii) Multiply $z$ by the *cholesky decomposition* $L$ of the correlation matrix $R$, i.e. $y = L z$,
        i.e. using the online matrix calculator [http://www.bluebit.gr/matrix-calculator].
    (iii) Transform $\{y_1, \ldots, y_n\}$, the components of $y$, into uniform random variates $\{u_1, \ldots, u_n\}$,
        i.e. using “NORMSINV( )”.
    (iv) Use the inverses of whatever marginals of $X_1, \ldots, X_n$ to transform $\{u_1, \ldots, u_n\}$ into $\{x_1, \ldots, x_n\}$.
    (v) Calculate $g(x_1, \ldots, x_n)$ and record/use the value for this one simulation run.
II. Methodology – pricing multiname (credit) derivatives

- Recall the *Equivalent Martingale Method (EMM)* of derivatives pricing.
  - Which essentially amounts to calculating the expectation of the final pay-off function, where expectation is taken w.r.t. the *risk neutral probability (measure)*.
  - With multiname (credit) derivatives, the expectation involve multiple underlyings, which in most cases are not statistically independent, hence a copula application.

- **Pricing vs. Calibration**
  - Recall how the *Black-Scholes (1973)* options pricing formula prices a European call on equity governed by *Geometric Brownian Motion (BGM)* using the $\sigma$ parameter; whereas, these days, the existence of a liquid options market allows us to “back out”, i.e. calibrate, the value of $\sigma$, hence “options-implied volatility”.
  - Assuming *homogeneous default correlation*, i.e. 1-parameter $R$, parameterised by a single $\rho$, with a liquid CDO market, i.e. CDX & iTraxx, one can also market calibrate the value for $\rho$, hence “compound/base correlation”.
III. Examples

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IV. Q&A (please!)

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